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A discrete element method for composite media: one-dimensional heat conduction

SHULING HOU and ALLEN C. COGLEY
Department of Mechanical Engineering, Kansas State University, Manhattan, KS 66506, U.S.A.

and

AJAY SHARMA†
IBM Corporation Research, Yorktown Heights, NY 10598, U.S.A.

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Abstract—Green's functions for steady-state, one-dimensional heat conduction for discretely inhomogeneous media are found by using an interactive principle. Exact solutions can therefore be expressed by Green's representation. An unsteady problem is solved numerically by treating the time derivative as a source term. The results demonstrate accuracy, stability, and efficiency.

INTRODUCTION

COMPOSITE materials are widely used. An example is large-scale integration of electrical circuits in computers and electronic devices. The high circuit density (which can be on the order of 2.5×10^7 per cubic meter) results in a serious heat conduction problem (with heat fluxes on the order of 100 kW m^{-2}) [1]. In computer module design, an efficient method is required to predict heat transfer fields in three-dimensional, discretely inhomogeneous media [1]. One-dimensional problems of heat conduction in layered solids can be solved analytically by the Laplace transform method [2, pp. 319–326]. A general integral transform technique can also be used to provide a systematic method for solving boundary-value problems of heat conduction in composite media [3, p. 594]. The final solution of temperature in this approach appears in the form of an infinite series in terms of the eigenfunctions. Although such solutions can be expressed analytically, they are difficult to evaluate since the eigenvalue problem is complex when the number of layers of the composite media is large. The Green's function method is another powerful tool for solving partial differential equations [3–7]. A Green's function formulation is presented in ref. [3, p. 604] for the temperature distribution in a layered material. A newly published book by J. V. Beck *et al.* [7] gives Green's function solutions for heat conduction with various boundary conditions. A Galerkin-based Green's function method for heat conduction with position-dependent coefficients is used to obtain an approximate solution [7, pp. 293–356]. The Green's functions used in refs. [3, 7] for layered media are local Green's functions that are matrices whose indices indicate the position of the source and the layer. In ref. [8] a global Green's function for an unsteady-state, N-layer structure is developed in Laplace transform space to solve an axisymmetric problem with volume heat sources.

This paper blends both analytic and numerical concepts into an efficient solution process for solving a prototype, one-dimensional, heat conduction problem for discretely inhomogeneous media. A simple form of the global Green's function for composite media in the steady-state case is found by making a change of variable and using the superposition principle. The temperature distribution is expressed directly by Green's representation.

To solve an unsteady problem, the power of Green's representation is used by treating the derivative of temperature with respect to time as a forcing function (inhomogeneous term). A finite difference scheme in time leads to discretized, simultaneous, algebraic equations for the unknown temperature at each point, including the interfaces. The solution is no longer analytic, but the analytical form of the solution is present through Green's-like matrices that show how the boundary condition affect the solution. Numerical results are presented for a specific problem. Extension to multidimensional media is required, but the concepts introduced here have applications to all physical phenomena supported on a continuous media.

GREEN'S REPRESENTATION FOR HOMOGENEOUS MEDIA

First consider a one-dimensional, steady-state heat conduction problem where the temperature T satisfies

$$\frac{d^2T}{dx^2} = -f(x)/k, \quad 0 < x < l. \tag{1}$$

† This paper is dedicated to Dr Ajay Sharma who lost his life in a tragic automobile accident during the work.

NOMENCLATURE

A	cross section area of a rod [m^2]
$\mathbf{A}$	matrix used for solving unsteady-state problem
A_1	coefficient in boundary conditions
A_2	coefficient in boundary conditions
$\mathbf{B}$	matrix used for solving unsteady-state problem
B_1	coefficient in boundary conditions
B_2	coefficient in boundary conditions
c	specific heat capacity [$W\ s\ kg^{-1}\ °C^{-1}$]
E	unit matrix
$f(x)$	internal source density [$W\ m^{-3}$]
f_0	reference source density [$W\ m^{-3}$]
G	Green's function
g_1	right hand side function in the left hand side boundary condition
g_2	right hand side function in the right hand side boundary condition
$H(x)$	Heaviside step function
k	thermal conductivity [$W\ m^{-1}\ °C^{-1}$]
l	length of a rod [m]
M	number of subintervals in a composite rod for numerical computation
$\mathbf{P}$	matrix used for solving unsteady-state problem
$\mathbf{Q}$	matrix used for solving unsteady-state problem
q	heat flux [$W\ m^{-2}$]
R	heat resistance [$°C\ W^{-1}$]
RL	total dimensionless heat resistance of composite rod [$°C\ W^{-1}$]
ΔR	the increment of R [$°C\ W^{-1}$]
T	temperature [°C]
T_1	boundary temperature at the left end [°C]
T_2	boundary temperature at the right end [°C]
$\tilde{T}_i$	dimensionless temperature at the interface
t	time [s]
t_0	reference time [s]
Δt	the increment of time [s]
x	physical space [m].

Greek symbols

α	derivative of the boundary temperature
δ	Delta function
κ	thermal diffusivity [$m^2\ s^{-1}$]
ξ	the position of the heat source [m]
ρ	density [$kg\ m^{-3}$]
σ	dimensionless parameters.

Subscripts

1	quantities in/for block 1
2	quantities in/for block 2
n	quantities in/for block n.

Superscripts

$\sim$	dimensionless quantities
$'$	local running variable in block 1
$''$	local running variable in block 2.

Here f is the internal source density and k the thermal conductivity. General boundary conditions are

$$A_1 T + B_1 \frac{dT}{dx} = g_1, \quad x = 0, \tag{2}$$

$$A_2 T + B_2 \frac{dT}{dx} = g_2, \quad x = l, \tag{3}$$

where the As and Bs are given constants. The Green's function, $G(x, \xi)$, is a field (response) at point x due to a point source (probe) of unit strength at point ξ under certain boundary and initial conditions. The Green's function corresponding to the above problem satisfies [5, p. 54]

$$\frac{\partial^2 G}{\partial x^2} = -\delta(x-\xi) \tag{4}$$

and the homogeneous boundary conditions

$$A_1 G + B_1 \frac{\partial G}{\partial x} = 0, \quad x = 0, \tag{5}$$

$$A_2 G + B_2 \frac{\partial G}{\partial x} = 0, \quad x = l. \tag{6}$$

Standard manipulations [7, p. 66] of this formulation leads to a solution in the form

$$T(x) = \int_0^l G(x,\xi)[f(\xi)/k]\,d\xi + \frac{1}{B_2} G|_{\xi=l} g_2 - \frac{1}{B_1} G|_{\xi=0} g_1. \tag{7}$$

Equation (7) is called Green's representation for Robin boundary conditions. Once the Green's function is known and the boundary conditions are given, the temperature distribution can be obtained from the Green's representation. The special cases of Dirichlet and Neumann boundary conditions can also be obtained.

GREEN'S REPRESENTATION IN INHOMOGENEOUS MEDIA

The Green's representation developed above is valid in a homogeneous medium. Extension to inhomogeneous media is required. Consider a composite rod consisting of two blocks with conductivities k_1

and k_2 as the model of an inhomogeneous region. At the interface, $x = x_1$, the heat flux is continuous as expressed by

$$k_1 \frac{dT}{dx}\bigg|_{x=x_1^-} = k_2 \frac{dT}{dx}\bigg|_{x=x_1^+}. \quad (8)$$

Equation (8) shows that the first derivative of temperature is not continuous at the interface. For composite material with perfect thermal contact the temperature is a single valued function of position, and the Green's function can be viewed as a temperature distribution caused by a unit source in block 1 or block 2. Hence, the Green's function itself is continuous everywhere, including the interface, but the first derivative of Green's function with respect to x has a jump at the interface. Consequently its second derivative at the interface becomes a δ-function just like the one at the source position. The Green's function would therefore satisfy

$$\frac{\partial^2 G}{\partial x^2} = -\delta(x-x') - \delta(x-\xi), \quad (9)$$

where x' locates the internal boundary. It can be shown that the temperature at the interface, which is unknown before the problem is solved, would be involved in the Green's representation if a Green's function is hypothesized for the composite medium. Therefore, the temperature cannot be obtained explicitly. To resolve this difficulty, the heat resistance, R, is introduced as a variable instead of x in the form

$$R = \int \frac{x}{kA}\,dx, \quad (10)$$

where A is the cross section area. This variable makes the first derivative of the Green's function continuous everywhere (except at the source). A reference quantity R_1, the total heat resistance in block 1, is chosen to normalize the global running variable $\tilde{R}$ such that

$$\tilde{R} = \begin{cases} \tilde{R}' = \dfrac{R'}{R_1} & 0 \leqslant \tilde{R} \leqslant 1 \\ 1 + \dfrac{R''}{R_1} = 1 + \tilde{R}'' & 1 \leqslant \tilde{R} \leqslant 1 + \dfrac{R_2}{R_1} \end{cases} \quad (11)$$

where R' and R'' are local dimensional running variables in block 1 and block 2, respectively. The quantities $\tilde{R}'$ and $\tilde{R}''$ are local dimensionless running variables. The dimensionless Green's function for a composite rod satisfies

$$\frac{\partial^2 \tilde{G}}{\partial \tilde{R}^2} = -\tilde{\delta}(\tilde{R}-\tilde{\xi}), \quad 0 < R, \xi < 1 + \frac{R_2}{R_1}. \quad (12)$$

This equation has the same form as that governing a homogeneous medium. Therefore, the Green's representation which is valid in a homogeneous domain can be extended to the inhomogeneous media. For heat conduction in a two-block composite rod, the dimensionless temperature satisfies

$$\frac{d^2\tilde{T}}{d\tilde{R}^2} = -\sigma_1 \tilde{f}_1, \quad 0 < \tilde{R} < 1, \quad (13)$$

$$\frac{d^2 T}{d\tilde{R}^2} = -\sigma_2 \tilde{f}_2, \quad 1 < \tilde{R} < 1 + \frac{R_2}{R_1}, \quad (14)$$

where $\sigma_1 = f_0 l_1^2/k_1 T_0$ and $\sigma_2 = f_0 l_2^2/k_2 T_0$ are dimensionless parameters and R_1 and R_2 are the total thermal resistance in block 1 and block 2, respectively. The Robin boundary conditions are

$$A_1\tilde{T} + B_1 \frac{d\tilde{T}}{d\tilde{R}} = \tilde{g}_1, \quad \tilde{R} = 0, \quad (15)$$

and

$$A_2\tilde{T} + B_2 \frac{d\tilde{T}}{d\tilde{R}} = \tilde{g}_2, \quad \tilde{R} = 1 + \frac{R_2}{R_1}. \quad (16)$$

The Green's representation takes the form

$$\tilde{T}(R) = \int_0^1 \sigma_1 \tilde{f}_1 \tilde{G}\,d\tilde{\xi} + \int_1^{1+(R_2/R_1)} \sigma_2 \tilde{f}_2 \tilde{G}\,d\tilde{\xi} + \frac{1}{B_2}\tilde{G}|_{\xi=1+(R_2/R_1)}\tilde{g}_2 - \frac{1}{B_1}\tilde{G}|_{\xi=0}\tilde{g}_1. \quad (17)$$

The Green's representation for any n-block composite rod can be written by changing the integral terms to n individual integrals over each block. Green's representations for Dirichlet and Neumann boundary conditions for an n-block composite medium can also be obtained.

GREEN'S FUNCTIONS FOR HOMOGENEOUS MEDIA

For heat conduction in a homogeneous medium, the dimensionless Green's function satisfies

$$\frac{\partial^2 \tilde{G}}{\partial \tilde{R}^2} = -\delta(\tilde{R}-\tilde{\xi}), \quad 0 < \tilde{R}, \tilde{\xi} < 1, \quad (18)$$

and the homogeneous Robin boundary conditions

$$A_1\tilde{G} + B_1 \frac{\partial \tilde{G}}{\partial \tilde{R}} = 0, \quad \tilde{R} = 0, \quad (19)$$

and

$$A_2\tilde{G} + B_2 \frac{\partial \tilde{G}}{\partial \tilde{R}} = 0, \quad \tilde{R} = l. \quad (20)$$

The Green's function is obtained by solving equation (18) directly [7, p. 478]. The solution can be written in the form

$$\tilde{G}(\tilde{R}, \tilde{\xi}) = \begin{cases} \dfrac{(A_1\tilde{R} - B_1)[A_2(1-\tilde{\xi}) + B_2]}{A_1A_2 + A_1B_2 - A_2B_1}, & 0 \leqslant \tilde{R} \leqslant \tilde{\xi}, \\ \dfrac{(A_1\tilde{\xi} - B_1)[A_2(1-\tilde{R}) + B_2]}{A_1A_2 + A_1B_2 - A_2B_1}, & \tilde{\xi} \leqslant \tilde{R} \leqslant 1. \end{cases} \quad (21)$$

The Green's function for Dirichlet boundary conditions

$$\tilde{G}(\tilde{R},\xi)|_{\tilde{R}=0} = 0, \quad \tilde{G}(\tilde{R},\xi)|_{\tilde{R}=1} = 0, \qquad (22)$$

can be easily obtained by setting $A_1 = 1$, $B_1 = 0$, $A_2 = 1$ and $B_2 = 0$ as follows:

$$\tilde{G}(\tilde{R},\xi) = \begin{cases} (1-\xi)\tilde{R}, & 0 \leqslant \tilde{R} \leqslant \xi, \\ (1-\tilde{R})\xi, & \xi \leqslant \tilde{R} \leqslant 1. \end{cases} \qquad (23)$$

By setting $A_1 = 1$, $B_1 = 0$, $A_2 = 0$, and $B_2 = 1$, the Green's function for the Neumann boundary conditions of

$$\tilde{G}(\tilde{R},\xi)|_{\tilde{R}=0} = 0, \quad \left.\frac{\partial \tilde{G}(\tilde{R},\xi)}{\partial \tilde{R}}\right|_{\tilde{R}=1} = 0, \qquad (24)$$

gives

$$\tilde{G}(\tilde{R},\xi) = \begin{cases} \tilde{R}, & 0 \leqslant \tilde{R} \leqslant \xi, \\ \xi, & \xi \leqslant \tilde{R} \leqslant 1. \end{cases} \qquad (25)$$

These Green's functions set the stage to obtain the Green's functions for composite media.

GREEN'S FUNCTIONS FOR COMPOSITE MEDIA

Heat conduction in a two-block composite rod with Robin boundary conditions is considered next. The Green's function satisfies

$$\frac{\partial^2 \tilde{G}}{\partial \tilde{R}^2} = -\delta(\tilde{R}-\xi), \quad 0 < \tilde{R},\xi < 1+\frac{R_2}{R_1}, \qquad (26)$$

and the homogeneous Robin boundary conditions are

$$A_1\tilde{G}+B_1\frac{\partial \tilde{G}}{\partial \tilde{R}} = 0, \quad \tilde{R} = 0, \qquad (27)$$

$$A_2\tilde{G}+B_2\frac{\partial \tilde{G}}{\partial \tilde{R}} = 0, \quad \tilde{R} = 1+\frac{R_2}{R_1}. \qquad (28)$$

Superposition can be applied to decompose the problem into three simpler problems shown in Fig. 1 (see ref. [9] for similar arguments). The first problem is the heat conduction in block 1 with a point source at ξ and boundary conditions

$$A_1\tilde{G}_1+B_1\frac{\partial \tilde{G}_1}{\partial \tilde{R}} = 0, \quad \tilde{R} = 0, \qquad (29)$$

$$\tilde{G}_1 = 0, \quad \tilde{R} = 1. \qquad (30)$$

The solution of this problem, $\tilde{G}_1$, can be obtained by setting $A_2 = 1$, $B_2 = 0$ in equation (21) to give

$$\tilde{G}_1(\tilde{R},\xi) = \begin{cases} \dfrac{(A_1\tilde{R}-B_1)(1-\xi)}{A_1-B_1} & 0 \leqslant \tilde{R} \leqslant \xi, \\ \dfrac{(A_1\xi-B_1)(1-\tilde{R})}{A_1-B_1} & \xi \leqslant \tilde{R} \leqslant 1. \end{cases} \qquad (31)$$

The second problem is the heat conduction in block 1 under the boundary condition

$$A_1\tilde{G}_2+B_1\frac{\partial \tilde{G}_2}{\partial \tilde{R}} = 0, \quad \tilde{R} = 0, \qquad (32)$$

$$\tilde{G}_2 = \tilde{T}_i \quad \tilde{R} = 1, \qquad (33)$$

where $\tilde{T}_i$ is the temperature at the interface. The solution of this problem, $\tilde{G}_2$, is found using Green's representation of equation (17) to be

$$\tilde{G}_2(\tilde{R},\xi) = \frac{A_1\tilde{R}-B_1}{A_1-B_1}\tilde{T}_i. \qquad (34)$$

The third problem is the heat conduction in block 2 with the boundary conditions given by

$$\tilde{G}_3 = \tilde{T}_i, \quad \tilde{R}'' = 0, \qquad (35)$$

$$A_2\tilde{G}_3+B_2\frac{\partial \tilde{G}_3}{\partial \tilde{R}} = 0, \quad \tilde{R}'' = \frac{R_2}{R_1}, \qquad (36)$$

where $\tilde{R}''$ is the local running variable in block 2. The solution of this problem, $\tilde{G}_3$, is again found through Green's representation to be

$$\tilde{G}_3(\tilde{R},\xi) = \left[1-\frac{A_2(\tilde{R}-1)}{A_2\dfrac{R_2}{R_1}+B_2}\right]\tilde{T}_i. \qquad (37)$$

The solution of the original problem, $\tilde{G}$, can therefore be written as

$$\tilde{G}(\tilde{R},\xi) = \begin{cases} \tilde{G}_1+\tilde{G}_2, & 0 \leqslant \tilde{R} \leqslant 1, \\ \tilde{G}_3, & 1 \leqslant \tilde{R} \leqslant 1+\dfrac{R_2}{R_1}. \end{cases} \qquad (38)$$

Substituting the expressions of $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$ into the equation above leads to

$$\tilde{G}(\tilde{R},\xi) = \begin{cases} \dfrac{(A_1\tilde{R}-B_1)(1-\xi)}{A_1-B_1}+\dfrac{(A_1\tilde{R}-B_1)}{A_1-B_1}\tilde{T}_i, \\ \qquad 0 \leqslant \tilde{R} \leqslant \xi, \\ \dfrac{(A_1\xi-B_1)(1-\tilde{R})}{A_1-B_1}+\dfrac{(A_1\tilde{R}-B_1)}{A_1-B_1}\tilde{T}_i, \\ \qquad \xi \leqslant \tilde{R} \leqslant 1, \\ \left(1-\dfrac{A_2(\tilde{R}-1)}{A_2\dfrac{R_2}{R_1}+B_2}\right)\tilde{T}_i, \\ \qquad 1 \leqslant \tilde{R} \leqslant 1+\dfrac{R_2}{R_1}. \end{cases} \qquad (39)$$

Matching the flux at the interface gives

$$\left.\frac{\partial \tilde{G}}{\partial \tilde{R}}\right|_{\tilde{R}=1^-} = \frac{-(A_1\xi-B_1)+A_1\tilde{T}_i}{A_1-B_1} = \frac{-A_2\tilde{T}_i}{A_2\dfrac{R_2}{R_1}+B_2} = \left.\frac{\partial \tilde{G}}{\partial \tilde{R}}\right|_{\tilde{R}=1^+}, \qquad (40)$$

which leads to the temperature at the interface, $\tilde{T}_i$, as

$$\tilde{T}_i = \frac{(A_1\xi-B_1)\left(A_2\dfrac{R_2}{R_1}+B_2\right)}{A_1A_2\left(1+\dfrac{R_2}{R_1}\right)+A_1B_2-A_2B_1}. \qquad (41)$$

Substituting $\tilde{T}_i$ into the expression of $\tilde{G}$ and doing

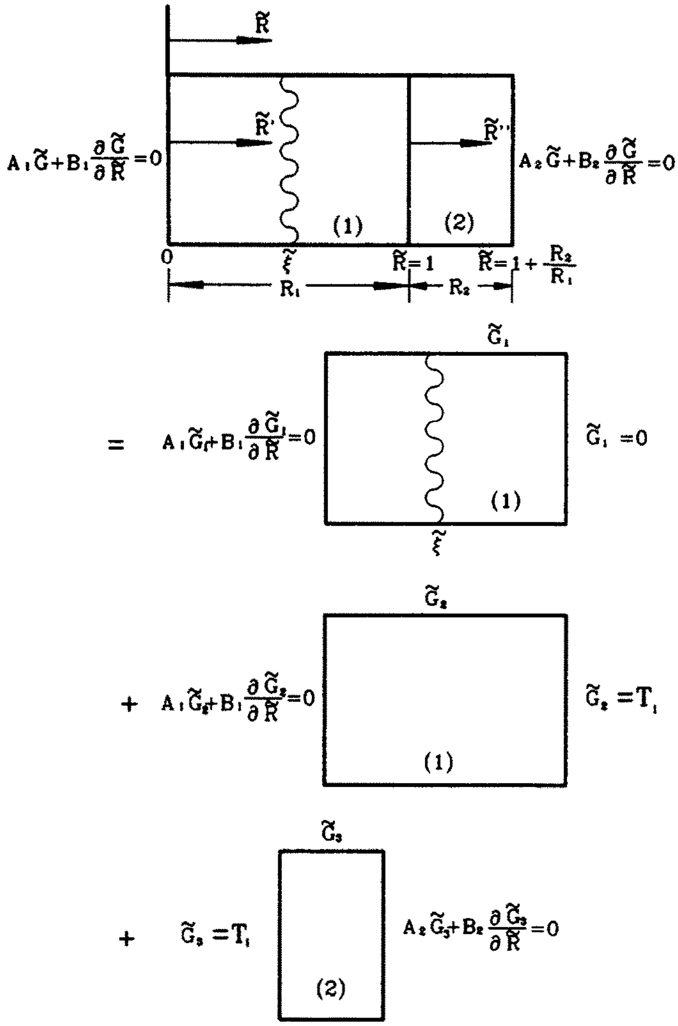

FIG. 1. Decomposition of two-block rod.

some algebra, the Green's function for the two-block composite media with Robin boundary conditions can be obtained as

$$\tilde{G}(\tilde{R},\tilde{\xi}) = \begin{cases} \dfrac{(A_1\tilde{R}-B_1)\left[A_2\left(1+\dfrac{R_2}{R_1}-\tilde{\xi}\right)+B_2\right]}{A_1A_2\left(1+\dfrac{R_2}{R_1}\right)+A_1B_2-A_2B_1}, & 0\leqslant\tilde{R}\leqslant\tilde{\xi}, \\[2ex] \dfrac{(A_1\tilde{\xi}-B_1)\left[A_2\left(1+\dfrac{R_2}{R_1}-\tilde{R}\right)+B_2\right]}{A_1A_2\left(1+\dfrac{R_2}{R_1}\right)+A_1B_2-A_2B_1}, & \tilde{\xi}\leqslant\tilde{R}\leqslant 1+\dfrac{R_2}{R_1}. \end{cases} \tag{42}$$

Following the same procedure, the Green's function for any n-block composite rod can be derived and takes the form

$$\tilde{G}(\tilde{R},\tilde{\xi}) = \begin{cases} \dfrac{(A_1\tilde{R}-B_1)\left[A_2\left(1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}-\tilde{\xi}\right)+B_2\right]}{A_1A_2\left(1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}\right)+A_1B_2-A_2B_1}, & 0\leqslant\tilde{R}\leqslant\tilde{\xi}, \\[2ex] \dfrac{(A_1\tilde{\xi}-B_1)\left[A_2\left(1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}-\tilde{R}\right)+B_2\right]}{A_1A_2\left(1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}\right)+A_1B_2-A_2B_1}, & \tilde{\xi}\leqslant\tilde{R}\leqslant 1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}. \end{cases} \tag{43}$$

Note that the expressions of Green's functions have only two formulations depending on the relative position of the source and the observation point, irrespective of which block contains the source.

Using appropriate values of A's and B's, the Green's function for an n-block composite rod subject to Dirichlet boundary conditions given by

$$\tilde{G}(\tilde{R},\tilde{\xi})|_{\tilde{R}=0}=0,\quad \tilde{G}(\tilde{R},\tilde{\xi})|_{\tilde{R}=1+(R_2/R_1)+\cdots+(R_n/R_1)}=0, \tag{44}$$

takes the form

$$\tilde{G}(\tilde{R},\tilde{\xi})=\begin{cases}\left(1-\dfrac{\tilde{\xi}}{1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}}\right)\tilde{R}, & \tilde{R}\leqslant\tilde{\xi},\\[2ex] \left(1-\dfrac{\tilde{R}}{1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}}\right)\tilde{\xi}, & \tilde{\xi}\leqslant\tilde{R}\leqslant 1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}.\end{cases} \tag{45}$$

For the Neumann boundary conditions expressed as

$$\tilde{G}(\tilde{R},\tilde{\xi})|_{\tilde{R}=0}=0,$$
$$\left.\frac{\partial\tilde{G}(\tilde{R},\tilde{\xi})}{\partial\tilde{R}}\right|_{\tilde{R}=1+(R_2/R_1)+\cdots+(R_n/R_1)}=0, \tag{46}$$

the Green's function is

$$\tilde{G}(\tilde{R},\tilde{\xi})=\begin{cases}\tilde{R}, & 0\leqslant\tilde{R}\leqslant\tilde{\xi},\\ \tilde{\xi}, & \tilde{\xi}\leqslant\tilde{R}\leqslant 1+\dfrac{R_2}{R_1}+\cdots+\dfrac{R_n}{R_1}.\end{cases} \tag{47}$$

Comparing these formulas with those for homogeneous media with $RL=1+(R_2/R_1)+\cdots+(R_n/R_1)$ as the total dimensionless thermal resistance of the composite rod, the formulations of the Green's function of composite media are structurally the same as those for homogeneous media with corresponding boundary conditions.

The steady-state heat conduction problem of a composite rod under different boundary conditions with constant heat generation can be solved analytically by solving the differential equations directly and implementing the boundary and interface conditions [10]. That solution can be used to check the present Green's function approach. The results are exactly the same. However, the present approach is valid and executable for arbitrary internal source distributions and any number of discrete blocks.

UNSTEADY-STATE CONDUCTION IN COMPOSITE RODS

The fact that a Green's function exists for composite media opens up unique solution approaches. Consider time dependent heat conduction in a homogeneous rod of length l. If R, the thermal resistance, is used as the independent variable the nondimensional governing equation is

$$\frac{\partial^2\tilde{T}}{\partial\tilde{R}^2}=\frac{l^2}{\kappa t_0}\frac{\partial\tilde{T}}{\partial\tilde{t}}=\sigma\frac{\partial\tilde{T}}{\partial\tilde{t}},\quad \tilde{t}>0,\ 0<\tilde{R}<1, \tag{48}$$

where $\kappa=k/\rho c$ is the thermal diffusivity. Here ρ is the density, c the specific heat and $\sigma=l^2/\kappa t_0$ a dimensionless parameter.

The unsteady Green's function for a homogeneous rod with temperature boundary conditions can be written in the form [4, p. 124]

$$\tilde{G}(\tilde{R},\tilde{\xi},\tilde{t},\tilde{\tau})=2H(\tilde{t}-\tilde{\tau})\times\sum_{n=1}^{\infty}\sin(n\pi\tilde{\xi})\sin(n\pi\tilde{R})\,\mathrm{e}^{-\sigma n^2\pi^2(\tilde{t}-\tilde{\tau})},$$

which is an infinite series involving four variables. Notice that in the unsteady case the temperature at the interface is no longer a constant but a function of time involving $\tilde{\xi}$ and $\tilde{\tau}$ as parameters. By using the procedure developed for steady-state problems, a Fredholm integral equation of the first kind with the temperature at the interface as the unknown function can be derived. The solution of this integral equation is not trivial. Moreover, the Green's Representation in this case is much more complex than for steady cases. It involves not only the convolution of the Green's function and initial or boundary condition terms, but also has integral terms involving temperature and Green's function and their derivatives evaluated at the interface. Solving unsteady heat conduction problems in composite rods by using unsteady Green's function will be complex. Based on this observation, a steady-state Green's function can be used to solve the unsteady problem by viewing the term $\sigma(\partial T/\partial t)$ in the governing equation as an internal source. For example, with Dirichlet boundary condition, the Green's Representation becomes

$$T(R,t)=\int_0^1-\sigma\dot{T}(\xi,t)G(R,\xi)\,\mathrm{d}\xi+T_1\left.\frac{\partial G}{\partial\xi}\right|_{\xi=0}-T_2\left.\frac{\partial G}{\partial\xi}\right|_{\xi=1}. \tag{49}$$

If heat generation also exists, the integral would include an additional source function. Using a backward differencing scheme to approximate $\dot{T}$ in equation (49), one obtains

$$T(R,t)=-\frac{\sigma}{\Delta t}\int_0^1 T(\xi,t)G(R,\xi)\,\mathrm{d}\xi+\frac{\sigma}{\Delta t}\int_0^1 T(\xi,t-\Delta t)G(R,\xi)\,\mathrm{d}\xi+T_1\left.\frac{\partial G}{\partial\xi}\right|_{\xi=0}-T_2\left.\frac{\partial G}{\partial\xi}\right|_{\xi=1}, \tag{50}$$

which is a Fredholm integral equation of the second kind for the unknown temperature $T(R,t)$.

The approach explained above is now used on an n-block composite rod. A quadrature procedure is performed to solve the integral equation and a set of algebraic equations is obtained in the matrix form

$$\mathbf{PT}(R, t_j) = \mathbf{QT}(R, t_{j-1}) + \mathbf{AB}, \tag{51}$$

where **B** is a matrix of boundary temperatures and **A** a matrix involving R_i. The matrices $\mathbf{T}(R, t_j)$ and $\mathbf{T}(R, t_{j-1})$ are the temperatures at each point and time levels j and $j-1$, respectively. The elements of matrix **Q** can be expressed as

$$Q_{ij}^k = \frac{\sigma_k \Delta\xi_k}{\Delta t}\left(1 - \frac{R_i}{RL}\right)\xi_j, \quad i > j, \tag{52}$$

and

$$Q_{ij}^k = \frac{\sigma_k \Delta\xi_k}{\Delta t}\left(1 - \frac{\xi_j}{RL}\right)R_i, \quad i < j, \tag{53}$$

where k indicates the number of block, and $i = 1, 2, \ldots, M-1$. The **P** matrix is given by $\mathbf{P} = \mathbf{E} + \mathbf{Q}$, where **E** is the unit matrix. The second term on the right hand side of equation (51) reflects the influence of the boundaries. The fact that the form of the Green's function shows up in the coefficient matrices **P** and **Q** reveals that the Green's function plays an important role in the algebraic system for unsteady problems. Similar results are obtained for the other boundary conditions.

NUMERICAL RESULTS

To verify the present solution for unsteady heat conduction in composite rods, an exact analytical solution for a two-block composite rod is solved by Laplace transforms [2]. A corresponding numerical model of a two-block composite rod is chosen (with $R_1 = 1.0$, $R_2 = 0.5$ and $\sigma_1 = 5.0$, $\sigma_2 = 1.0$). Blocks 1 and 2 are divided into 100 and 50 subintervals, respectively, so $\Delta R_1 = \Delta R_2 = 0.01$. Figure 2 shows a comparison between the exact and DEM solutions for temperature distribution vs R for the two different times of $t = 0.5$ and $t = 1.0$. The time step is $\Delta t = 0.025$. The plot shows that the two solutions are very close when $t = 0.5$. When $t = 1.0$, the difference between these two solutions is not detectable on this figure.

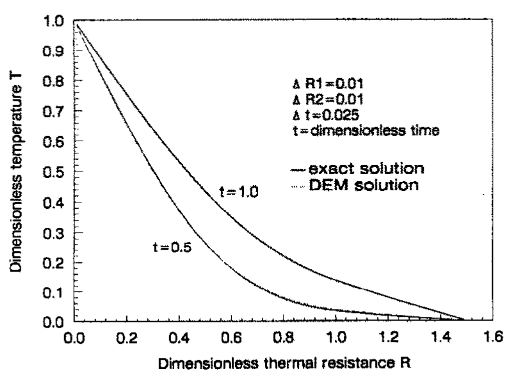

FIG. 2. Comparison between exact and DEM solution for two-block rod.

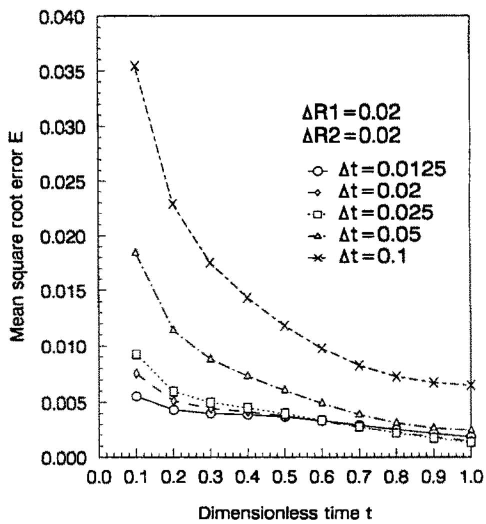

FIG. 3. MSR error vs time for two-block model.

To study the properties of the numerical solution, the mean square root error (denoted by MSR error) is calculated for many cases. Figure 3 is a plot for a given ΔR of 0.02. There are five curves corresponding to $\Delta t = 0.0125, 0.020, 0.025, 0.050, 0.100$, respectively. The MSR error depends on t, ΔR, and Δt and is a decreasing function of t. For the case of $\Delta R = \Delta t$ and both are less than 0.025, the MSR error is within the range of 0.001 to 0.01 for $t > 0.1$. This is quite acceptable.

To show all the capabilities of the present formulation, a five-block composite rod subject to Dirichlet boundary conditions is solved. The numerical steps are chosen as $\Delta R = 0.01$ in each block and $\Delta t = 0.025$. Figure 4 shows the results for $t = 0.2$, 0.4, 0.6, 0.8, 1.0. Figure 5 shows the same results vs x, the physical space, for the same conditions but later time $t = 4.0$, 6.0, 8.0, 10.0 to see the tendency to approach the

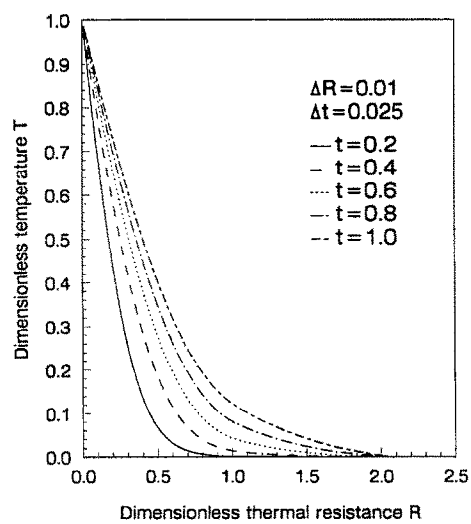

FIG. 4. Temperature vs R for five-block model.

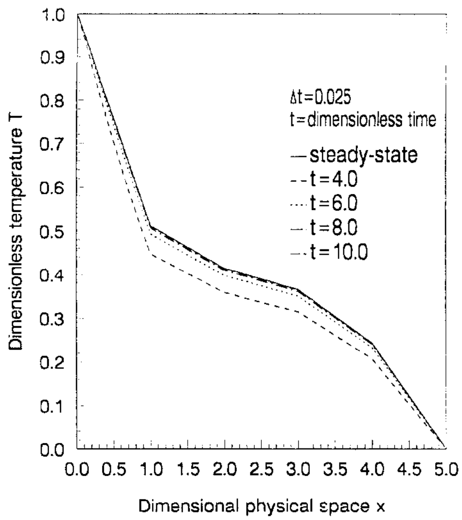

FIG. 5. Unsteady DEM and analytical steady solution.

steady-state. In this physical variable the slope of the temperature is not continuous. Since no exact solution for this case is available, a comparison between the unsteady and the corresponding steady-state solution given by the solid line is made in Fig. 5. These results show that the unsteady solution approaches the steady-state solution asymptotically.

The last example is a square plate with unit sides made of two different materials. Each region has a width 0.5 m, with the left hand side region having $k = 1.0$ and the right hand side region having $k = 10.0$. The two lateral surfaces are insulated such that the exact solution (for steady-state) can be obtained. The other two sides designated a and b are subject to a prescribed temperature, $T_a = 100°C$, and heat flux $q_b = 50\ \mathrm{W\ m^{-2}}$, respectively. Comparisons between the results from the present method, boundary element method and the exact steady-state analysis [11] is given as follows:

Exact	DEM	BEM
$\Delta T_i = 25.00°C$	$\Delta T_i = 24.96°C$	$\Delta T_i = 24.72°C$,
$\Delta T_b = 27.50°C$	$\Delta T_b = 27.46°C$	$\Delta T_b = 27.21°C$,

where ΔT_i and ΔT_b are the temperature increases at the interface and the right hand boundary, respectively, based on the temperature at the left hand boundary. The relative error for temperature at the interface is 0.16% for the DEM and 1.12% for the BEM.

CONCLUSIONS

The preceding work shows that the Green's functions method is useful in solving one-dimensional heat conduction problems in discretely inhomogeneous media. By stretching the space variable, a steady-state heat conduction problem defined for inhomogeneous rods can be converted to the same problem in homogeneous rods. Therefore, the Green's Representation can be used for inhomogeneous regions and the temperature expressed in the terms of Green's function. The Green's function used in this mapped space is a single function and can be obtained analytically by the superposition procedure for various boundary conditions. The Green's function defined on the original physical space is a function matrix that can be constructed in terms of eigenfunctions [3, p. 604]. Since the eigenvalues for an inhomogeneous medium are not easy to obtain, the present approach simplifies the problem. For steady-state heat conduction problems in composite rods, this method gives analytical solutions that are easier to evaluate for the temperature, compared with other analytical methods.

For time dependent problems, steady-state composite-media Green's functions and a finite difference scheme can be used to reduce the partial differential equation to a set of algebraic equations. The Green's function and boundary condition relationships are retained in the resulting algebraic equation. Computational examples show that this method produces accurate solution when compared with certain simple existing exact solutions. The numerical program for any n-block composite rod is easy to implement and the computation is stable. The error analysis confirmed the concept that minimum error is obtained when $\Delta t \approx \Delta R$.

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